Hierarchy and Anarchy in Quark Mass Matrices, or Can Hierarchy Tolerate Anarchy? [1]

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Abstract

The consequences of adding random perturbations (anarchy) to a baseline hierarchical model of quark masses and mixings are explored. Even small perturbations of the order of 5% of the smallest non-zero element can already give deviations significantly affecting parameters of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, so any process generating the anarchy should in general be limited to this order of magnitude. The regularities of quark masses and mixings thus appear to be far from a generic feature of randomness in the mass matrices, and more likely indicate an underlying order.

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1 Introduction

The origin of fermion masses and mixings is one of most important issues in particle physics. Unfortunately, these parameters are inputs in the well-tested Standard Model. All one can do is to measure them as accurately as possible and hope that in the future a more fundamental theory will actually be able to predict their values. Many attempts have been made in this direction, with grand unified theories (whether supersymmetric or not) being favorite candidates [1]. However, such a fundamental theory could well be quite complicated, with many new fields and couplings. Consequently, at low energies, the observed fermion masses and mixings could be the result of a large number of contributions. If that were the case, one might expect that at low energy scales the mass matrices of fermions would have a random nature. This idea was first suggested by Froggatt and Nielsen [2], who performed a statistical study of fermion masses without success.

More recently, the idea of flavor anarchy was introduced in order to explain new data on neutrino masses and mixings [3], and several analyses based on this idea have been performed [4]. It was suggested that a similar model could also explain masses and mixings in the quark and charged lepton sectors. The purpose of this note is to study the robustness of such a model for random quark matrices in the quark sector. We take a baseline model which approximately reproduces the observed masses and mixings and study its sensitivity to random perturbations of the parameters. In this way we determine to what extent the observed regularities can survive effects which may be purely coincidental or generic. We find that quark masses and mixings appear to be far from a generic feature of randomness in the mass matrices, and more likely point to an underlying order.

2 A simple ansatz

The quark and charged lepton sectors are fundamentally different from the neutrino sector because of the existence of a large mass hierarchy between the families and the resulting small mixing. We shall construct a "baseline" description of quark masses and mixings which incorporates several approximate regularities. For this purpose we begin with quark masses evolved via the renormalization group to a common high mass scale $\mu = M_Z$. At this scale, the masses have been found to lie in the range [5]

$$\begin{aligned}
m_{\rm u}(M_{\rm Z}) &= 0.9 - 2.9 \ \text{MeV} \\
m_{\rm c}(M_{\rm Z}) &= 0.53 - 0.68 \ \text{GeV} \\
m_{\rm t}(M_{\rm Z}) &= 168 - 180 \ \text{GeV} \\
m_{\rm d}(M_{\rm Z}) &= 1.8 - 5.3 \ \text{MeV} \\
m_{\rm s}(M_{\rm Z}) &= 35 - 100 \ \text{MeV} \\
m_{\rm b}(M_{\rm Z}) &= 2.8 - 3.0 \ \text{GeV}
\end{aligned} \tag{1}$$

with corresponding ratios

$$\sqrt{\frac{m_{\rm u}}{m_{\rm c}}} = \frac{1}{28} - \frac{1}{14}~; \quad \sqrt{\frac{m_{\rm c}}{m_{\rm t}}} = \frac{1}{18} - \frac{1}{16}~;$$

$$\sqrt{\frac{m_{\rm d}}{m_{\rm s}}} = \frac{1}{7.4} - \frac{1}{2.6}~; \quad \sqrt{\frac{m_{\rm s}}{m_{\rm b}}} = \frac{1}{9.1} - \frac{1}{5.3}~. \tag{2}$$

These masses thus are compatible with the hierarchy

$$\frac{m_2}{m_3} = \frac{m_1}{m_2} = \epsilon^2~, \tag{3}$$

which we shall incorporate into our *ansatz* for quark mass matrices. (This is certainly not a property of the charged leptons, for which [6]

$$\frac{m_2/3}{m_3} = \frac{m_1}{m_2/3} = \epsilon^2 \tag{4}$$

is a better approximation.) For illustrative purposes we will adopt $\epsilon_{\rm up} \equiv \epsilon_{\rm u} = 0.07$ and $\epsilon_{\rm down} \equiv \epsilon_{\rm d} = 0.21$, which approximately reproduces the observed hierarchies.

We also seek a set of mass matrices reproducing the regularities

$$|V_{us}| \simeq |V_{cd}| \simeq \mathcal{O}\left(\sqrt{\frac{m_d}{m_s}}\right) = \epsilon_{\rm d} \quad , \quad |V_{cb}| \simeq |V_{ts}| \simeq \mathcal{O}\left(\frac{m_s}{m_b}\right) = \epsilon_{\rm d}^2 \quad . \tag{5}$$

The first of these was noted some time ago [7, 8, 9]. We further wish to reproduce the hierarchy of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements noted by Wolfenstein [10], in which

$$|V_{us}| \simeq |V_{cd}| \simeq \mathcal{O}(\lambda)~, \quad |V_{cb}| \simeq |V_{ts}| \simeq \mathcal{O}(\lambda^2)~, \quad |V_{ub}| \simeq |V_{td}| \simeq \mathcal{O}(\lambda^3)~, \tag{6}$$

($\lambda \simeq 0.22$), and with $|V_{ub}| < |V_{td}|$ as favored by fits to data [11].

These regularities can be incorporated into a simple quark mass ansatz in a basis which we call *hierarchical*:

$$\mathcal{M}_H ~=~ m_3 \left(\begin{array}{ccc} 0 & \epsilon^3 e^{i\phi} & 0 \\ \epsilon^3 e^{-i\phi} & \epsilon^2 & \epsilon^2 \\ 0 & \epsilon^2 & 1 \end{array} \right)~, \tag{7}$$

where m_3 denotes the mass eigenvalue of the third-family quark (t or b). Hierarchical descriptions of this type were first introduced by Froggatt and Nielsen [12]. The present ansatz is closely related to one described by Fritzsch and Xing [13, 14]. We shall not be concerned with relative coefficients of order 1 in different terms; for example, models in which the off-diagonal ϵ^2 terms are multiplied by $\sqrt{2}$ may fit $|V_{cb}|$ somewhat better [13, 15]. The ϵ^3 terms in our model are separate parameters in Ref. [14]. We have assumed the phase to be present only in the ϵ^3 terms; there is little sensitivity to phases in the off-diagonal ϵ^2 terms [14]. Our purpose is primarily

to construct an easily-manipulated "cartoon" version of the mass matrices, so as to study the robustness of their predictions for masses and mixings under random perturbations.

The eigenvalues of the matrix (7) to order ϵ^4 are given by

$$\lambda_1 = -m_3\epsilon^4 \tag{8}$$

$$\lambda_2 = m_3\epsilon^2 \tag{9}$$

$$\lambda_3 = m_3(1+\epsilon^4) \tag{10}$$

and are independent of the phase ϕ. Therefore, in this ansatz the quark masses naturally obey the hierarchy (3).

The matrix (7) can be made real by a unitary transformation $\mathcal{M}_{\rm H}^{\rm R} = P\mathcal{M}_{\rm H}P^\dagger$, where P is given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix} . \tag{11}$$

The real symmetric matrix $\mathcal{M}_{\rm H}^{\rm R}$ then can be diagonalized by an orthogonal transformation $O^T\mathcal{M}_{\rm H}^{\rm R}O$ where the matrix O is given to order ϵ^4 (its columns are u, c, t) by:

$$O = \begin{pmatrix} 1-\frac{\epsilon^2}{2}+\frac{3\epsilon^4}{8} & \epsilon-\frac{\epsilon^3}{2} & 0 \\ -\epsilon+\frac{\epsilon^3}{2} & 1-\frac{\epsilon^2}{2}-\frac{\epsilon^4}{8} & \epsilon^2+\frac{\epsilon^4}{2} \\ \epsilon^3 & -\epsilon^2-\epsilon^4 & 1-\frac{\epsilon^4}{2} \end{pmatrix} . \tag{12}$$

We will assume this same mass matrix ansatz for both the up and the down quark sectors. In this case, the Cabibbo-Kobayashi-Maskawa (CKM) matrix is readily obtained by:

$$V_{\rm CKM} = O_{\rm up}^T P_{\rm up} P_{\rm down}^\dagger O_{\rm down} \quad . \tag{13}$$

We first give approximate expressions for the CKM matrix elements:

$$V_{\rm ud} = 1-(\epsilon_{\rm u}^2+\epsilon_{\rm d}^2)/2+e^{i\Delta}\epsilon_{\rm u}\epsilon_{\rm d}+\mathcal{O}(\epsilon^4) \tag{14}$$

$$V_{\rm us} = \epsilon_{\rm d}-e^{i\Delta}\epsilon_{\rm u}+\mathcal{O}(\epsilon^3) \tag{15}$$

$$V_{\rm ub} = \epsilon_{\rm u}e^{i\Delta}(\epsilon_{\rm u}^2-\epsilon_{\rm d}^2)+\mathcal{O}(\epsilon^5) \tag{16}$$

$$V_{\rm cd} = \epsilon_{\rm u}-e^{i\Delta}\epsilon_{\rm d}+\mathcal{O}(\epsilon^3) \tag{17}$$

$$V_{\rm cs} = e^{i\Delta}[1-(\epsilon_{\rm d}^2+\epsilon_{\rm u}^2)/2]+\epsilon_{\rm u}\epsilon_{\rm d}+\mathcal{O}(\epsilon^4) \tag{18}$$

$$V_{\rm cb} = e^{i\Delta}(\epsilon_{\rm d}^2-\epsilon_{\rm u}^2)+\mathcal{O}(\epsilon^4) \tag{19}$$

$$V_{\rm td} = \epsilon_{\rm d}e^{i\Delta}(\epsilon_{\rm d}^2-\epsilon_{\rm u}^2)+\mathcal{O}(\epsilon^5) \tag{20}$$

$$V_{\rm ts} = e^{i\Delta}(\epsilon_{\rm u}^2-\epsilon_{\rm d}^2)+\mathcal{O}(\epsilon^4) \tag{21}$$

$$V_{\rm tb} = e^{i\Delta}+\mathcal{O}(\epsilon^4) \; , \tag{22}$$

where $\Delta \equiv \phi_{\rm u}-\phi_{\rm d}$. These can be brought into a form closer to the standard phase convention (see, e.g., [10]) by multiplying the c and t rows by $e^{i(\chi-\Delta)}$ and the s and b

columns by $e^{-i\chi}$, where $\chi = \mathrm{Arg}(\epsilon_{\rm d} - e^{i\Delta}\epsilon_{\rm u})$ is chosen so as to make V_{us} and V_{cd} real. Then we find, to leading order in small terms,

$$\begin{aligned}
V_{\rm ud} &= 1 - (\epsilon_{\rm u}^2 + \epsilon_{\rm d}^2)/2 + e^{i\Delta}\epsilon_{\rm u}\epsilon_{\rm d} && (23)\\
V_{\rm us} &= |\epsilon_{\rm d} - e^{i\Delta}\epsilon_{\rm u}| && (24)\\
V_{\rm ub} &= \epsilon_{\rm u} e^{i(\Delta-\chi)}(\epsilon_{\rm u}^2 - \epsilon_{\rm d}^2) && (25)\\
V_{\rm cd} &= -|\epsilon_{\rm d} - e^{i\Delta}\epsilon_{\rm u}| && (26)\\
V_{\rm cs} &= 1 - (\epsilon_{\rm u}^2 + \epsilon_{\rm d}^2)/2 + e^{-i\Delta}\epsilon_{\rm u}\epsilon_{\rm d} && (27)\\
V_{\rm cb} &= \epsilon_{\rm d}^2 - \epsilon_{\rm u}^2 && (28)\\
V_{\rm td} &= \epsilon_{\rm d} e^{i\chi}(\epsilon_{\rm d}^2 - \epsilon_{\rm u}^2) && (29)\\
V_{\rm ts} &= \epsilon_{\rm u}^2 - \epsilon_{\rm d}^2 && (30)\\
V_{\rm tb} &= 1\ . && (31)
\end{aligned}$$

The angles in the unitarity triangle can be expressed very simply in terms of these quantities. We find

$$\alpha(=\phi_2) = \Delta \quad , \quad \beta(=\phi_1) = -\chi = \tan^{-1}\left(\frac{\sin\Delta}{\epsilon_{\rm d}/\epsilon_{\rm u} - \cos\Delta}\right) \quad ,$$

$$\gamma(=\phi_3) = \pi - \alpha - \beta \quad . \tag{32}$$

These expressions also hold in more general versions of the present model [13, 14]. Note that $|V_{ub}|$ and $|V_{td}|$ are specified entirely in terms of the $\epsilon_{\rm u,d}$, with

$$|V_{cb}| = |V_{ts}| = \epsilon_{\rm d}^2 - \epsilon_{\rm u}^2 = 0.0392 \quad , \quad |V_{ub}/V_{td}| = \epsilon_{\rm u}/\epsilon_{\rm d} = 1/3 \tag{33}$$

for our choice of parameters. The shape of the unitarity triangle is determined entirely by the magnitude of V_{us}, which changes as Δ varies. It has been noticed previously that a value of Δ close to $90°$ gives a good fit to $|V_{us}|$ [8, 9]. For $\Delta = 90°$ we find $|V_{us}| = |V_{cd}| = (\epsilon_{\rm d}^2 + \epsilon_{\rm u}^2)^{1/2} = 0.221$. The Wolfenstein parameters ρ and η defined by

$$\begin{aligned}
\lambda\sqrt{\rho^2 + \eta^2} &= \tfrac{|V_{\rm ub}|}{|V_{\rm cb}|} = \epsilon_{\rm u} && (34)\\
\lambda\sqrt{(1/c^2 - \rho)^2 + \eta^2} &= \tfrac{|V_{\rm td}|}{|V_{\rm ts}|} = \epsilon_{\rm d}\ , && (35)
\end{aligned}$$

where $\lambda = |V_{\rm us}|$ and $c = 1 - \lambda^2/2$, are given by:

$$\begin{aligned}
\rho &= \frac{1}{2c}\left(1 - \frac{c^2}{\lambda^2}(\epsilon_{\rm d}^2 - \epsilon_{\rm u}^2)\right) && (36)\\
\eta &= \sqrt{\frac{\epsilon_{\rm u}^2}{\lambda^2} - \rho^2}\ . && (37)
\end{aligned}$$

With our choice of parameters, $\rho = 0.12$ and $\eta = 0.29$.

We now diagonalize the u and d quark mass matrices exactly (without the above approximations), and calculate the resulting CKM matrix. As an illustration, for the same parameters as above, we find

$$|V_{\rm ud}| = |V_{\rm cs}| = 0.975 \quad (38)$$
$$|V_{\rm us}| = |V_{\rm cd}| = 0.222 \quad (39)$$
$$|V_{\rm cb}| = |V_{\rm ts}| = 0.0410 \quad (40)$$
$$|V_{\rm td}| = 0.0080\ ;\ \ |V_{\rm ub}| = 0.0029\ ;\ \ |V_{\rm ub}/V_{\rm cb}| = 0.071 \quad (41)$$
$$\alpha = 98^\circ\ ;\ \ \beta = 18^\circ\ [\sin(2\beta) = 0.58]\ ;\ \ \gamma = 64^\circ \quad (42)$$
$$\rho = 0.15\ ;\ \ \eta = 0.29\quad . \quad (43)$$

These predictions are reasonable and we will consider this simple model as our baseline model. We now perform random perturbations around it (anarchy) to study the robustness of these predictions.

3 Anarchy

The underlying theory of fermion masses is unlikely to guarantee that a particular mass matrix element is absolutely zero or has a fixed value given by high energy symmetries of the theory. One could expect radiative effects or small symmetry breaking parameters to contribute to a given mass matrix texture. Since we don't have a complete theory to compute them, we will explore the consequences of adding randomly small perturbations to our baseline model. These perturbations are what we call anarchy. We will study just how much anarchy can our baseline model tolerate.

We will add to each element of the up and down mass matrices a random variable ζ with real part given by a random gaussian of zero average and a phase given by a random variable uniformly distributed in the range $[0, 2\pi]$. We ensure that the mass matrices remain hermitian by requiring that $\zeta_{ij} = \zeta^*_{ji}$.

The standard deviation of the real part of these random variables will be our measure of anarchy. In fact, it would be natural to have the standard deviations proportional to the smallest non-zero elements of the mass matrix, namely $\sigma = \alpha\epsilon^3$, where σ is the standard deviation of the gaussian distribution.

In Figs. 1, 2, and 3 we illustrate the effect of anarchy in the $\rho - \eta$ plane and the $\frac{m_2}{m_1} - \frac{m_3}{m_2}$ planes for both up and down sectors for different values of α. As expected, the dispersion in the figures increases with α. The quantities ρ, η and $\frac{m_3}{m_2}$ are more stable under perturbations than the ratio of the two lightest quark masses, $\frac{m_2}{m_1}$.

In order to measure the degree of variation of these parameters arising from anarchy, we introduce the quantity κ, the ratio of the standard deviation σ_X to the mean μ_X for a given parameter X in a simulation: $\kappa \equiv \sigma_X/\mu_X$. In Table 1 we show κ for the different parameters generated by 1000 simulations.

We can see that κ is roughly proportional to the parameter α determining the width of the gaussian distribution used to generate the different perturbations. The sensitivity in the parameters increases in the following order: $\left(\frac{m_2}{m_3}\right)_{\rm up}$, $\left(\frac{m_2}{m_3}\right)_{\rm down}$, η,

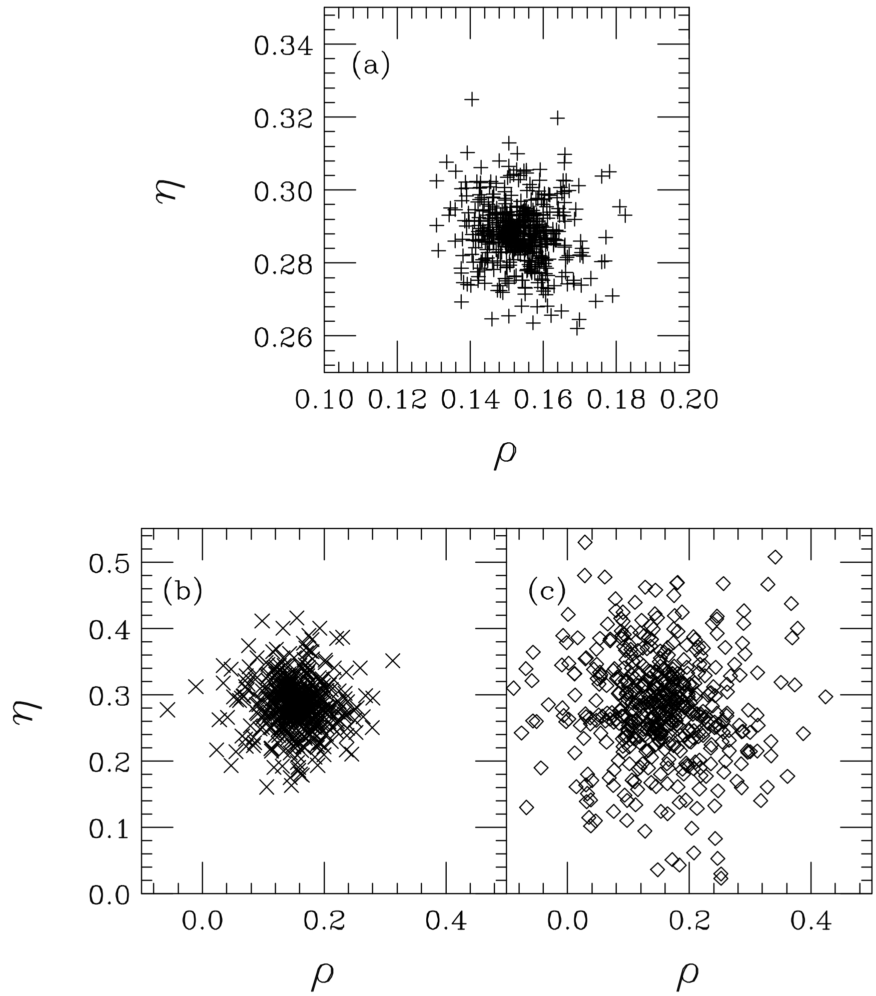
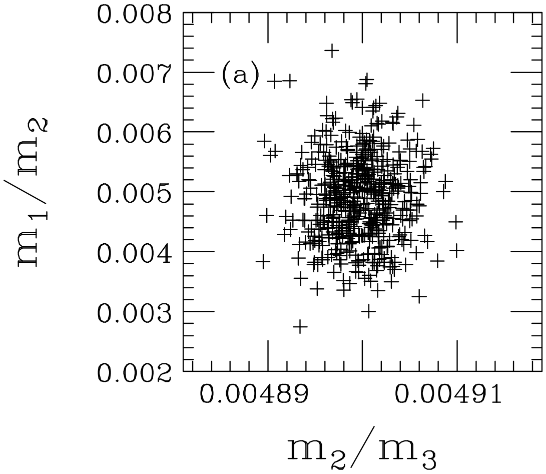


Figure 1: Effect of anarchy in the $\rho - \eta$ plane for different values of α. (a) $\alpha = 0.01$; (b) $\alpha = 0.05$; (c) $\alpha = 0.1$.

$\left(\frac{m_1}{m_2}\right)_{\text{down}}$, ρ and $\left(\frac{m_1}{m_2}\right)_{\text{up}}$. Not surprisingly, the most sensitive parameter is $\left(\frac{m_1}{m_2}\right)_{\text{up}}$, since it involves the ratio of two small quantities. Values of $\alpha < 0.1$ are required in order not to disturb the hierarchy significantly. These correspond roughly to $\mathcal{O}(\epsilon_d^5) \simeq \mathcal{O}(\lambda^5)$ entries in the mass matrices (7).

4 Conclusions

We have performed a simple exercise of exploring the consequences of adding random perturbations (anarchy) to a baseline hierarchical model of quark masses and mixings. We find that even small perturbations, with gaussian distribution of zero mean and standard deviations of the order of 5% of the smallest non-zero element can already give deviations of 26% in the ρ parameter, for instance. Therefore, we conclude that any physics process generating the anarchy, be it radiative corrections



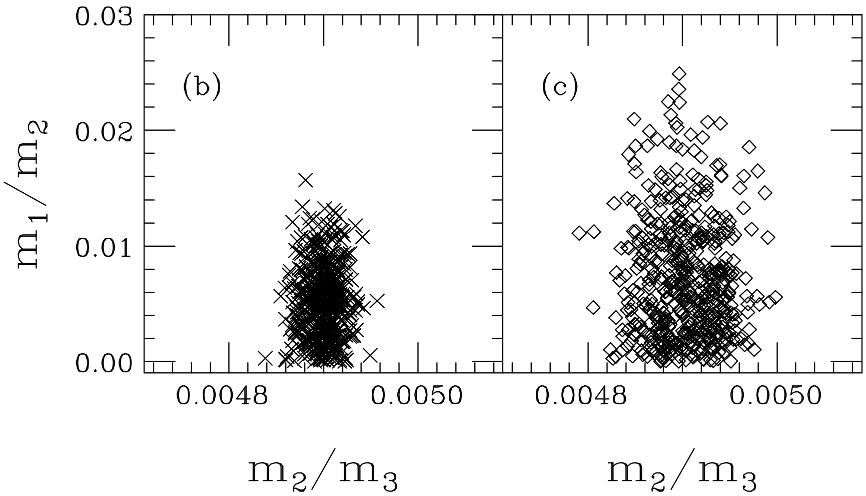


Figure 2: Effect of anarchy in the up sector in the $\frac{m_1}{m_2} - \frac{m_2}{m_3}$ plane for different values of α. (a) $\alpha = 0.01$; (b) $\alpha = 0.05$; (c) $\alpha = 0.1$.

Table 1: Ratios κ of the standard deviations to the means for parameters generated by 1000 simulations.

	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
ρ	0.054	0.26	0.57
η	0.029	0.14	0.28
$\left(\frac{m_2}{m_3}\right)_{\rm up}$	0.00072	0.0034	0.0071
$\left(\frac{m_1}{m_2}\right)_{\rm up}$	0.14	0.60	0.73
$\left(\frac{m_2}{m_3}\right)_{\rm down}$	0.0021	0.010	0.021
$\left(\frac{m_1}{m_2}\right)_{\rm down}$	0.046	0.24	0.45

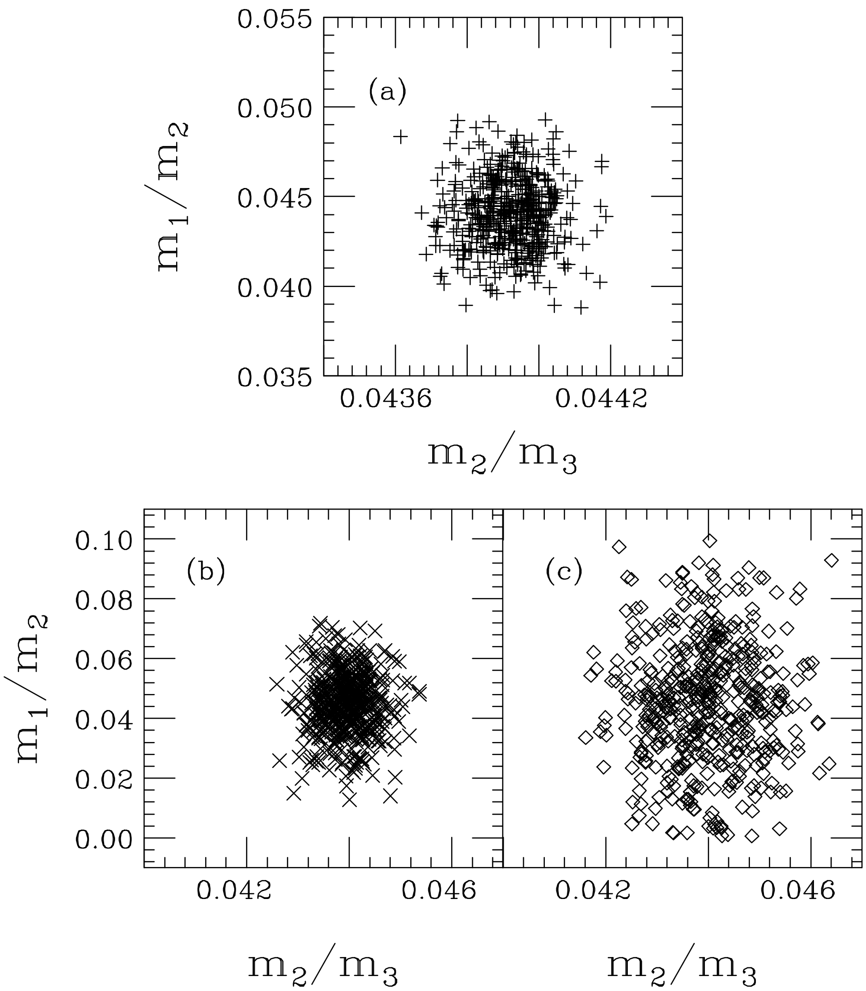


Figure 3: Effect of anarchy in the down sector in the $\frac{m_1}{m_2} - \frac{m_2}{m_3}$ plane for different values of α. (a) $\alpha = 0.01$; (b) $\alpha = 0.05$; (c) $\alpha = 0.1$.

or small symmetry breaking parameters, should be in general limited to this order of magnitude, unless some spurious cancellations occur. The regularities of quark masses and mixings thus appear to be far from a generic feature of randomness in the mass matrices, and probably indicative of an underlying order.

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